Optimal cloning of coherent states by linear optics

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We describe an optical scheme for optimal Gaussian $n \to m$ cloning of coherent states. The scheme, which generalizes a recently demonstrated scheme for $1 \to 2$ cloning, involves only linear optical components and homodyne detection.

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I. INTRODUCTION

The generation of perfect copies of a given, unknown, quantum state is impossible [1, 2, 3, 4]. Analogously, starting from n copies of a given, unknown, quantum state no device exists that provides $m > n$ perfect copies of those states. On the other hand, one can make approximate copies of quantum states by means of a quantum cloning machine [5], whose performances may be assessed by the *single clone fidelity*, namely, a measure of the similarity between each of the clones and the input state. A cloner is said to be *universal* if the fidelity is independent on the input state, whereas the cloning process is said to be *optimal* if the fidelity saturate an upper bound $F^{(\mathrm{opt})}$, which depends on the class of states under investigation, as well as on the class of involved operations. For coherent states and Gaussian cloning (i.e., cloning by Gaussian operations) $F^{(\mathrm{opt})} = 2/3$ whereas, using non-Gaussian operations, it is possible to achieve $F \approx 0.6826 > 2/3$ [6]. Therefore, though non-Gaussian operations are of some interest [7, 8, 9, 10], the realization of optimal Gaussian cloning would provide performances not too far from the ultimate bound imposed by quantum mechanics.

Optimal Gaussian cloning of coherent states may be implemented using an appropriate combination of beam splitters and a single phase insensitive parametric amplifier [11, 12]. However, the implementation of an efficient phase insensitive amplifier operating at the fundamental limit is still a challenging task. This problem was solved by Andersen et al. [13], who proposed and experimentally realized an optimal cloning machine for coherent states, which relies only on linear optical components and a feed-forward loop [14]. As a consequence of the simplicity and the high quality of the optical devices used in this experiment, performances close to optimal ones were attained. The thorough theoretical description of this cloning machine as well as its average fidelity for different ensembles of input states has been given in [15], and a generalization to asymmetric cloning was presented in [16]

In this paper we describe in details a generalization of the cloning machine considered in [13] to realize $n \to m$ universal cloning of coherent states. The scheme involves only linear optical components and homodyne detection and yields the optimal cloning fidelity [17]. Analogue schemes has been proposed for broadcasting a complex amplitude bby Gaussian states [18].

The paper is structured as follows: in section II we described the linear cloning machine for $1 \to m$ cloning of coherent states and we give the conditions to achieve universal and optimal cloning as in the case of $1 \to 2$. In section III we deal with a scheme to realize $n \to m$ optimal universal cloning. Finally, in section IV we draw some concluding remarks.

II. THE $1 \to m$ CLONING MACHINE

The scheme of the $1 \to m$ Gaussian cloning machine is sketched in Fig. 1. The coherent input state $|\alpha\rangle$ is mixed with the vacuum at a beam splitter (BS) with transmissivity τ. On the reflected part, double-homodyne detection is

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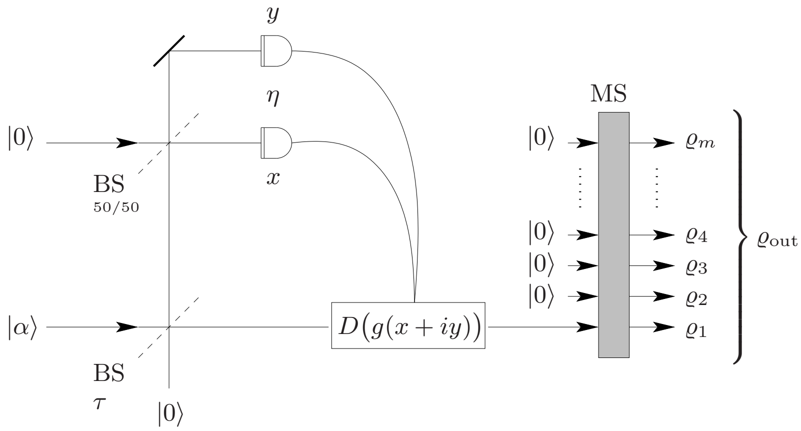

FIG. 1: Gaussian cloning of coherent states by linear optics: the input state $|\alpha\rangle$ is mixed with the vacuum $|0\rangle$ at a beam splitter (BS) of transmissivity τ. The reflected beam is measured by double-homodyne detection and the outcome of the measurement $x + iy$ is forwarded to a modulator, which imposes a displacement $g(x + iy)$ on the transmitted beam, g being a suitable amplification factor. Finally, the displaced state is impinged onto a multi-splitter (MS), where it is mixed with $m-1$ vacuum modes. The states ϱ_k, $k = 1, m$, are the m clones.

performed using two detectors with equal quantum efficiencies η: this measurement is executed by splitting the state at a balanced beam splitter and, then, measuring the two conjugate quadratures $\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$ and $\hat{y} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^\dagger)$, with $\hat{a}$ and $\hat{a}^\dagger$ being the field annihilation and creation operator. The outcome of the double-homodyne detector gives the complex number $z = x + iy$. According to these outcomes, the transmitted part of the input state undergoes a displacement by an amount gz, where g is a suitable electronic amplification factor. Finally, the m output states, denoted by the density operators ϱ_k, $k = 1, \ldots, m$, are obtained by dividing the displaced state using a multi-splitter (MS). When $m = 2$ the present scheme reduces to a $1 \to 2$ Gaussian cloning machine recently experimentally realized [13] and studied in details [15].

If we denote with U_τ the evolution operator of the first BS with transmissivity τ, after the BS we have:

$$U_\tau \,|\alpha\rangle \otimes |0\rangle = \left|\alpha\sqrt{\tau}\right\rangle \otimes \left|\alpha\sqrt{1-\tau}\right\rangle \; ; \tag{1}$$

the reflected beam, i.e., $\left|\alpha\sqrt{1-\tau}\right\rangle$ undergoes a double-homodyne detection described by the positive operator-valued measure (POVM) [19]

$$\Pi_\eta(z) = \int_{\mathbb{C}} d^2\zeta \, \frac{1}{\pi\sigma_\eta^2} \exp\left\{-\frac{|\zeta - z|^2}{\sigma_\eta^2}\right\} \frac{|\zeta\rangle\langle\zeta|}{\pi} \,, \tag{2}$$

with $\sigma_\eta^2 = (1-\eta)/\eta$, η being the detection quantum efficiency, and, in turn, the probability of getting z as outcome is given by:

$$p_\eta(z) = \mathrm{Tr}[\Pi_\eta(z) \left|\alpha\sqrt{1-\tau}\right\rangle \left\langle\alpha\sqrt{1-\tau}\right|] \tag{3}$$

$$= \frac{\eta}{\pi} \exp\left\{-\eta|z - \alpha\sqrt{1-\tau}|^2\right\} \, . \tag{4}$$

After the measurement, the transmitted part of the input state, i.e., $|\alpha\sqrt{\tau}\rangle$, is displaced by an amount gz, and, averaging over all the possible outcomes z, we obtain the following state:

$$\varrho = \int_{\mathbb{C}} d^2z \, p_\eta(z) \, D(gz) \left|\alpha\sqrt{\tau}\right\rangle \left\langle\alpha\sqrt{\tau}\right| D^\dagger(gz) \tag{5}$$

$$= \int_{\mathbb{C}} d^2z \, \frac{\eta}{\pi} \exp\left\{-\eta|z - \alpha\sqrt{1-\tau}|^2\right\} \left|\alpha\sqrt{\tau} + gz\right\rangle \left\langle\alpha\sqrt{\tau} + gz\right| \, , \tag{6}$$

which is then mixed in the MS with $m-1$ vacuum modes (Fig. 1). The MS acts on a single coherent state $|\beta\rangle$ as follows:

$$U_{\mathrm{MS}} \,|\beta\rangle_1 \otimes |0\rangle_2 \otimes \ldots \otimes |0\rangle_m = \left|\frac{\beta}{\sqrt{m}}\right\rangle_1 \otimes \left|\frac{\beta}{\sqrt{m}}\right\rangle_2 \otimes \ldots \otimes \left|\frac{\beta}{\sqrt{m}}\right\rangle_m \,, \tag{7}$$

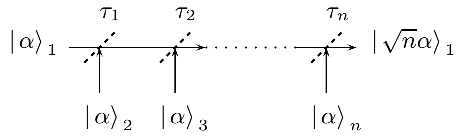

FIG. 2: Cascade of BSs with transmissivity $\tau_k = (1+k)^{-1}$: the n-mode input state $|\Psi\rangle_{\rm in} = \otimes_{k=1}^{n} |\alpha\rangle_k$ is converted into the output $|\Psi\rangle_{\rm out} = |\sqrt{n}\alpha\rangle_1 \otimes_{k=2}^{n} |0\rangle_k$.

where the subscripts refer to the mode entering the MS and $U_{\rm MS}$ being the MS evolution operator [20]. In turn, the m-mode state emerging from the MS reads:

$$\varrho_{\rm out} = \int_{\mathbb{C}} d^2z\, p_\eta(z) \bigotimes_{k=1}^{m} \left| \frac{\alpha\sqrt{\tau}+gz}{\sqrt{m}} \right\rangle_{k\,k} \left\langle \frac{\alpha\sqrt{\tau}+gz}{\sqrt{m}} \right| . \tag{8}$$

Note that, in practice, the average over all possible outcomes z in Eq. (5) should be performed at this stage, that is after the MS. On the other hand, because of the linearity of the integration, the results are identical, but performing the averaging just before the MS simplifies the calculations. Moreover, notice also that the m-mode state (8) is *separable* and all the m outputs ϱ_k are equal.

As figure of merit to characterize the performance of the $1 \to m$ cloning machine, we consider the fidelity, which is a measure of similarity between the hypothetically perfect clone, i.e., the input state, and the actual clone. If the cloning fidelity is independent on the initial state the machine is referred to as a *universal* cloner. In the present case, the fidelity is the same for all the clones ϱ_k and is given by:

$$F_\eta(\alpha,\tau,m) = \langle\alpha|\, \varrho_k\, |\alpha\rangle \tag{9}$$

$$= \int_{\mathbb{C}} d^2z\, p_\eta(z) \exp\left\{ -\left| \alpha - \frac{\alpha\sqrt{\tau}+gz}{\sqrt{m}} \right|^2 \right\} \tag{10}$$

$$= \frac{m\eta}{g^2+m\eta} \exp\left\{ -\frac{\eta\left[g\sqrt{1-\tau}+\sqrt{\tau}-\sqrt{m}\right]^2}{g^2+m\eta} |\alpha|^2 \right\} . \tag{11}$$

If we set

$$g = \frac{\sqrt{m}-\sqrt{\tau}}{\sqrt{1-\tau}} , \tag{12}$$

Eq. (11) becomes independent on the input coherent state amplitude, i.e., we have an universal cloning machine, and we get:

$$F_\eta(\tau,m) = \frac{m\eta(1-\tau)}{(\sqrt{m}-\sqrt{\tau})^2 + m\eta(1-\tau)} , \tag{13}$$

which reaches its maximum

$$F_\eta^{(\max)}(m) = \frac{m\eta}{(1+\eta)m-1} , \tag{14}$$

when $\tau = 1/m$. Notice that if $\eta \to 1$, then one obtains the optimal $1 \to m$ cloning fidelity, i.e.,

$$\lim_{\eta\to 1} F_\eta^{(\max)}(m) = \frac{m}{2m-1} \equiv F_{1\to m}^{(\rm opt)} . \tag{15}$$

III. $n \to m$ CLONING

The linear cloning machine can be also used to produce m copies of n equal input coherent states ($m > n$). Given two coherent states, $|\alpha\rangle_1$ and $|\sqrt{k}\alpha\rangle_2$, one has

$$U_k\, |\alpha\rangle_1 \otimes |\sqrt{k}\alpha\rangle_2 = |\sqrt{k+1}\alpha\rangle_1 \otimes |0\rangle_2 , \tag{16}$$

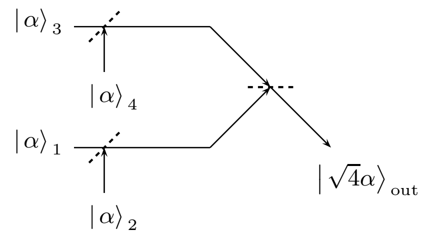

FIG. 3: Simplified scheme able to convert 4 coherent states with the same amplitude α into a single coherent state with amplitude $\sqrt{4}\alpha$. All the involved BSs are balanced and the scheme can be easily extended to the case of 2^k input states and $2^k - 1$ BSs.

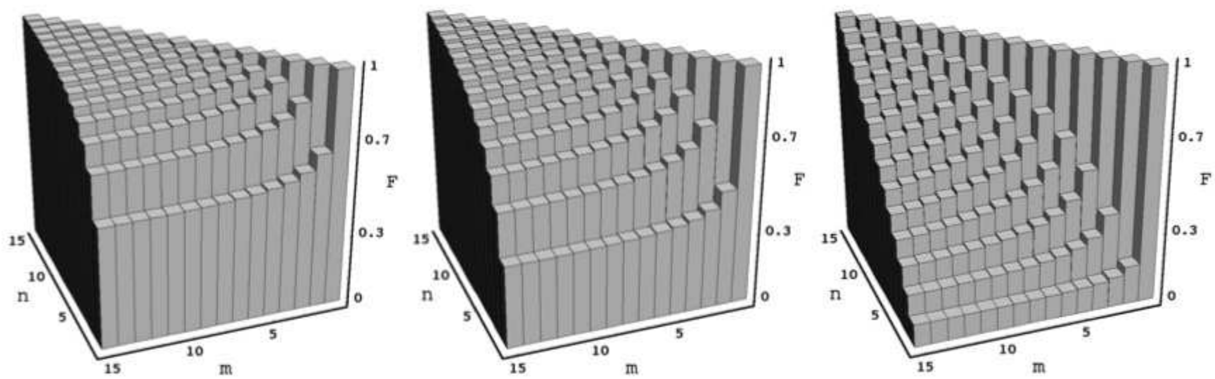

FIG. 4: Fidelity of optimal $n \to m$ cloning of coherent states as a function of the number of input (n) and output ($m \geq n$) copies for different values of the quantum efficiency (from left to right $\eta = 0.99, 0.5, 0.1$).

U_k being the evolution operator associated with a BS with transmissivity $\tau_k = (1+k)^{-1}$; in turn, using a suitable cascade of BSs, we can transform the n-mode input state $|\Psi\rangle_{\text{in}} = \otimes_{k=1}^{n} |\alpha\rangle_k$ into the output $|\Psi\rangle_{\text{out}} = |\sqrt{n}\alpha\rangle_1 \otimes_{k=2}^{n} |0\rangle_k$ (see Fig. 2) [12]. This scheme becomes very simple if $n = 2^k$: in this case one only needs $n-1$ *balanced* BSs to produce $|\Psi\rangle_{\text{out}}$, as depicted in Fig. 3 for $n = 4$. Now, we take $|\sqrt{n}\alpha\rangle$ as input state of the $1 \to m$ cloning machine described above obtaining the following new expression for the fidelity

$$F_\eta(\alpha, \tau, n, m) = \frac{m\eta}{g^2 + m\eta} \exp\left\{ -\frac{\eta \left[g\sqrt{n(1-\tau)} + \sqrt{n\tau} - \sqrt{m} \right]^2}{g^2 + m\eta} |\alpha|^2 \right\}, \tag{17}$$

which becomes independent on the amplitude α (universal cloning) if

$$g = \frac{\sqrt{m} - \sqrt{n\tau}}{\sqrt{n(1-\tau)}}, \tag{18}$$

and reaches its maximum

$$F_\eta^{(\max)}(n, m) = \frac{mn\eta}{mn\eta + m - n}, \tag{19}$$

when $\tau = n/m$ (see Fig. 4). As in the case of $1 \to m$ cloning, if $\eta \to 1$ then we obtain

$$\lim_{\eta \to 1} F_\eta^{(\max)}(n, m) = \frac{mn}{mn + m - n} \equiv F_{n \to m}^{(\text{opt})}, \tag{20}$$

i.e., the maximum fidelity achievable in $n \to m$ cloning [21].

IV. CONCLUSIONS

We have addressed the $1 \to m$ and the $n \to m$ Gaussian cloning of coherent states based on an extension of a linear $1 \to 2$ cloning machine, namely, a cloner which relies only on linear optical elements and a feed-forward loop. In both

$1 \rightarrow m$ and $n \rightarrow m$ cloning, we have shown that the electronic gain and the BS transmissivity can be chosen in such a way that the machine acts as an optimal universal Gaussian cloner. We can conclude that the linear cloning machine represents a highly versatile tool.

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